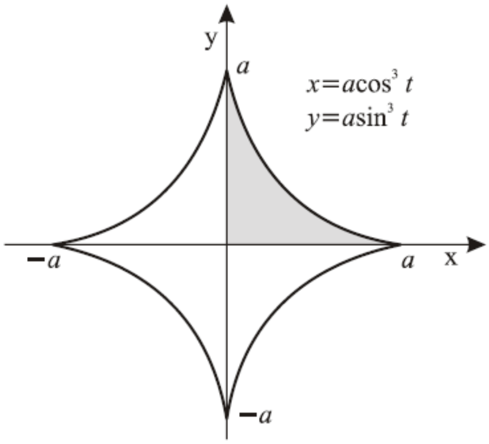
1. (20) Odredite površinu omeđenu astroidom $x^{2/3}+y^{2/3}=a^{2/3}$.



Rješenje:

Ako $x^{2/3}+y^{2/3}=a^{2/3}$ zapišemo u obliku $\left(x^{\frac{1}{3}}\right)^2+\left(y^{\frac{1}{3}}\right)^2=\left(a^{\frac{1}{3}}\right)^2$, lako uočavamo kako $\left(x^{\frac{1}{3}},y^{\frac{1}{3}}\right)$ leže na kružnici radijusa $a^{\frac{1}{3}}$. Stoga ih vrlo lako možemo prikazati parametarski

$$\begin{matrix} x^{\frac{1}{3}}=a^{\frac{1}{3}}\cos\mathrm{t} \\ y^{\frac{1}{3}}=a^{\frac{1}{3}}\sin\mathrm{t}\end{matrix} \quad\Rightarrow\quad \begin{matrix} x=a\cos^3\mathrm{t} \\ y=a\sin^3\mathrm{t}\end{matrix} \tag{1}$$

Kako je astroida simetrična izračunat ćemo četvrtinu površine (S, osjenčeni dio, između osi x i astroide).
Granice:

$$\begin{matrix} x=0, \mathrm{y}=\mathrm{a} \\ x=a, y=0\end{matrix} \overset{(1)}{\Rightarrow} \begin{matrix} t=\pi/2 \\ t=0\end{matrix}$$

Četvrtina površine iznosi

$$S=\int_S dS=\int_S dxdy=\int_0^a dx\int_0^y dy=\int_0^a ydx=\begin{Bmatrix} dx=-3a\cdot\cos^2\mathrm{t}\cdot\sin\mathrm{t}\cdot\mathrm{dt} \\ y=a\sin^3\mathrm{t}\end{Bmatrix}=-3a^2\int_{\pi/2}^{0}\cos^2\mathrm{t}\cdot\sin^4\mathrm{t}\cdot\mathrm{dt}$$

$$S=3a^2\int_0^{\pi/2}\sin^4\mathrm{t}\cdot\cos^2\mathrm{t}\cdot\mathrm{dt}=3a^2\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)^2\cdot\frac{1+\cos 2t}{2}\cdot\mathrm{dt}=\frac{3a^2}{8}\int_0^{\pi/2}(1-\cos 2t)(1-\cos^2 2t)\,\mathrm{dt}$$

$$S=\frac{3a^2}{8}\int_0^{\pi/2}(1-\cos 2t-\cos^2 2t+\cos^3 2t)\,\mathrm{dt}=\frac{3a^2}{8}(S_1-S_2-S_3+S_4)$$

$$S_1=\int_0^{\pi/2}\mathrm{dt}=t\Big|_0^{\pi/2}=\frac{\pi}{2}-0=\frac{\pi}{2}$$

$$S_2=\int_0^{\pi/2}\cos(2t)\,\mathrm{dt}=\frac{1}{2}\int_0^{\pi/2}\cos 2t\,\mathrm{d(2t)}=-\frac{1}{2}\sin 2t\Big|_0^{\pi/2}=-\frac{1}{2}\cdot\sin\pi+\frac{1}{2}\cdot\sin 0=-0+0=0$$

$$S_3=\int_0^{\pi/2}\cos^2(2t)\,\mathrm{dt}=\int_0^{\pi/2}\frac{1+\cos 4t}{2}\mathrm{dt}=\frac{1}{2}\left(\left(t+\frac{1}{4}\sin(4t)\right)\Big|_0^{\pi/2}\right)=\frac{1}{2}\left(\frac{\pi}{2}+\frac{1}{4}\sin(2\pi)-0-\frac{\sin(0)}{4}\right)=\frac{\pi}{4}$$

$$S_4=\int_0^{\pi/2}\cos^3(2t)\,\mathrm{dt}=\int_0^{\pi/2}\cos^2 2t\cdot\cos 2\mathrm{t}\cdot\mathrm{dt}=\frac{1}{2}\int_0^{\pi/2}(1-\sin^2 2\mathrm{t})\mathrm{d}(\sin 2\mathrm{t})=\frac{1}{2}\left(\sin 2\mathrm{t}-\frac{\sin^3 2\mathrm{t}}{3}\right)\Big|_0^{\pi/2}=0$$

$$S=\frac{3a^2}{8}(S_1-S_2-S_3+S_4)=\frac{3a^2}{8}\left(\frac{\pi}{2}-0-\frac{\pi}{4}-0\right)=\frac{3a^2\pi}{4\cdot 8}\xrightarrow{UKUPNA\ \ POVRŠINA}P=4\cdot S=\frac{3a^2\pi}{8}$$